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New examples of zero modes

Daniel M Elton

School of Mathematical Sciences, University of Sussex, Falmer, Brighton BN1 9QH, UK

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Abstract. We construct smooth vector potentials $A^{(1)}$ and $A^{(2)}$ on $\mathbb{R}^3$ such that the associated Pauli operators $\sigma \cdot (D - A^{(i)})$, $i = 1, 2$, have zero-energy eigenfunctions (or *zero modes*). The first example $A^{(1)}$ has compact support and the associated zero mode is contained in L^2. The second example can be written in the form $A^{(2)} = (-x_2, x_1, 0)^{\mathrm{T}} + \tilde{A}^{(2)}$ where $\tilde{A}^{(2)}$ is bounded and supported on $\{|x_3| \leqslant 1\}$. Furthermore the associated zero mode is Schwartz class.

1. Introduction

This paper is concerned with the kernel of the Pauli operator $\sigma \cdot (D - A)$ on $\mathbb{R}^3$. Here $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ where the σ_is are the Pauli matrices, $D = -\mathrm{i}\nabla$ where $\nabla = (\partial_1, \partial_2, \partial_3)$ denotes the gradient operator on $\mathbb{R}^3$ and A is a real vector (magnetic) potential. The Pauli operator acts on two-component spinor fields, which (for our purposes) are simply $\mathbb{C}^2$-valued functions on $\mathbb{R}^3$. We are interested in studying vector potentials for which the associated Pauli operator possesses zero-energy eigenfunctions or *zero modes*; that is, we wish to study those A for which there exists a nontrivial spinor field $\psi \in L^2$ such that $\sigma \cdot (D - A)\psi = 0$.

Apart from their intrinsic interest, the study of zero modes has considerable importance in mathematical physics. The first examples of zero modes were given in [9] where the authors were principally motivated by an accompanying paper [6]. In the latter it was proved that the existence of zero modes of the Pauli operator on $\mathbb{R}^3$ implied the collapse of single-electron atoms for sufficiently large nuclear charge.

A second area in which zero modes arise is in the study of the three-dimensional fermionic determinant in quantum electrodynamics. The existence and degeneracy of zero modes is related to the nonperturbative behaviour of this determinant for massive fermions in strong magnetic fields; see [7, 8] for further details.

Paper [9] contains not only the first explicit examples of zero modes but two general methods for their construction (appearing in sections 2 and 3, respectively). The explicit examples in [9], together with further examples given in [1], all come from applications of the method in section 2. The original examples of [9] were studied in greater detail in [2], where it was shown that the set of zero modes for a given magnetic potential could be arbitrarily degenerate. The authors of [1, 2] also observed relationships between their examples, a certain 'topological number' and Hopf maps (see also [3]). The latter appeared in a related manner in [5]; here a new class of examples was constructed, essentially by pulling back particular magnetic fields from $\mathbb{R}^2$ to $\mathbb{R}^3$ via stereographic projections and Hopf maps. These fields were then shown to produce zero modes related to the well known Aharonov–Casher zero modes (see [4]) of the original two-dimensional fields.

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In [8] comparisons between zero modes in two, three and four dimensions were made. In particular it was suggested that zero modes in three dimensions may have a topological origin, as is the case for the Aharonov–Casher zero modes (their two-dimensional counterparts). The classes of examples mentioned above, together with the associated observations, tend to support such an idea. However, it should be emphasized that the study of zero modes in three dimensions is still in its infancy and no general results pertaining to a topological link exist at present.

This paper concerns the construction of two new examples of zero modes in three dimensions. We begin by constructing smooth magnetic potentials $A^{(1)}$ and $A^{(2)}$ on $\mathbb{R}^3$ and then use ideas from section 3 of [9] to find spinor fields $\psi^{(1)}$ and $\psi^{(2)}$ which satisfy $\sigma \cdot (D - A^{(i)})\psi^{(i)} = 0$ for $i = 1, 2$. Particular properties of these examples are as follows.

Example 1. The spinor field $\psi^{(1)}$ is contained in the Sobolev space $H^{p,s}$ for any $p \in (3/2, +\infty]$ and $s \in \mathbb{R}$. The magnetic potential $A^{(1)}$ is smooth and is supported on the unit ball in $\mathbb{R}^3$. It follows that the associated magnetic field $B^{(1)} := \mathrm{curl} A^{(1)}$ is also smooth and supported on the unit ball.

Example 2. The spinor field $\psi^{(2)}$ is Schwartz class; that is, $\psi^{(2)}$ is smooth and, along with derivatives of arbitrary order, has super-power decay as $|x| \to +\infty$. We can write the magnetic potential in the form $A^{(2)} = (-x_2, x_1, 0)^{\mathrm{T}} + \tilde{A}^{(2)}$ where $\tilde{A}^{(2)}$ is smooth, bounded and supported on $\{|x_3| \leqslant 1\}$. The associated magnetic field $B^{(2)} := \mathrm{curl} A^{(2)}$ can then be written as a perturbation of the constant magnetic field $(0, 0, 2)^{\mathrm{T}}$ by $\tilde{B}^{(2)} := \mathrm{curl} \tilde{A}^{(2)}$. This perturbation is smooth, supported on $\{|x_3| \leqslant 1\}$ and decays like $\mathrm{O}(|x|^{-1})$ as $|x| \to +\infty$.

These examples do not appear to be included in the classes of examples produced in [1,2,5] or section 2 of [9]. Furthermore they appear to be the first explicit examples employing the general method for constructing zero modes as outlined in section 3 of [9]. In the absence of general results about three-dimensional zero modes it is important to broaden the class of known examples as much as possible. In particular, new examples are useful for determining which topological properties are specific to a given class of examples and which might be present in general.

These new examples are also important for restricting the type of analytic conditions on a magnetic potential that might ensure the nonexistence of zero modes. For example, the arguments in section 4 of [6] show that any unidirectional magnetic field cannot possess zero modes (note: only the direction is assumed to be fixed; the field strength can vary). Furthermore a classical particle in a magnetic field which is constant in a conical region of $\mathbb{R}^3$ with axis parallel to the field is free to escape to infinity along the field lines. It is then not unreasonable to conjecture that the corresponding quantum system (i.e. the Pauli equation) will not admit zero-energy bound states (i.e. zero modes). Example 2 obviously shows this conjecture to be false. In fact, the two examples clearly impose very broad limitations on the type of analytic conditions that might guarantee the nonexistence of zero modes. In this respect these examples lend weight to the idea that zero modes in three dimensions have a topological origin.

Notation. Points in $\mathbb{R}^3$ are denoted as $x = (x_1, x_2, x_3)$. We also use radial coordinates r and ρ on $\mathbb{R}^3$ defined by

$$r = |x| = (x_1^2 + x_2^2 + x_3^2)^{1/2} \qquad \text{and} \qquad \rho = (x_1^2 + x_2^2)^{1/2}.$$

If u is a function on $\mathbb{R}^+$ then $u(r)$ defines a radially symmetric function on $\mathbb{R}^3$. Whenever the domain is clear from the context this latter function is also denoted by u. In general, if u is smooth on $\mathbb{R}^+$ and has a Taylor series expansion which is valid at 0 and contains only even powers, then $u(r)$ is smooth on $\mathbb{R}^3$.

We use ∂_ρ to denote the differential operator on $\mathbb{R}^3$ defined by $\rho\partial_\rho = x_1\partial_1 + x_2\partial_2$, whilst a prime denotes differentiation on $\mathbb{R}$ or $\mathbb{R}^+$. Thus, with u as given above, $\partial_i u = x_i r^{-1} u'$.

2. Auxiliary results

The construction of both examples relies on the 'if' part of the next result (which is a summary of observations made in the first part of section 3 of [9]).

Proposition 1. *Suppose ψ is a smooth spinor field on $\mathbb{R}^3$ with $\langle\psi, \psi\rangle$ nowhere vanishing and A is a real vector potential also on $\mathbb{R}^3$. Then ψ and A satisfy*

$$\sigma \cdot (D - A)\psi = 0$$

if and only if they satisfy the pair of equations

$$A = \frac{1}{\langle\psi, \psi\rangle}\left(\frac{1}{2}\mathrm{curl}\langle\psi, \sigma\psi\rangle + \mathrm{Im}\,\langle\psi, \nabla\psi\rangle\right) \qquad \textit{and} \qquad \mathrm{div}\langle\psi, \sigma\psi\rangle = 0.$$

Remark. For any spinor $\psi = \binom{\psi_1}{\psi_2}$ we have

$$\langle\psi, \sigma\psi\rangle = \begin{pmatrix} 2\mathrm{Re}\,(\overline{\psi_1}\psi_2) \\ 2\mathrm{Im}\,(\overline{\psi_1}\psi_2) \\ |\psi_1|^2 - |\psi_2|^2 \end{pmatrix}. \tag{1}$$

The construction of both examples makes use of a common auxiliary function. For the remainder of the paper we suppose g is a fixed choice of function with the following properties:

(A1) $g : \mathbb{R} \to \mathbb{R}$ is smooth, compactly supported and non-negative.
(A2) $g(t) = (4 - t^2)^{1/2}$ for $t \in [-1/2, 1/2]$.
(A3) $\mathrm{supp}(g) \subseteq [-1, 1]$.
(A4) $\pm g'(t) \leqslant 0$ for $\pm t \geqslant 0$.

Together, properties (A1)–(A4) imply $\mathrm{Ran}(g) = [0, 2]$. Also property (A2) ensures the radially symmetric function $g(r)$ is smooth on $\mathbb{R}^3$.

3. Example 1

Define a real valued function f on $\mathbb{R}^+$ by

$$f(r) = r^{-3}\left(-\int_0^r t^4 (g^2)'(t)\,\mathrm{d}t\right)^{1/2} \tag{2}$$

for all $r > 0$. Using (A1)–(A4) it can be seen that the integrand is smooth, nonpositive for any $t \geqslant 0$ and strictly negative for sufficiently small $t > 0$. Therefore the integral is strictly negative for any $r > 0$. It follows that f is smooth on $\mathbb{R}^+$. Also, using (A2), we get $f(r) = 1/\sqrt{3}$ for $r \in (0, 1/2]$. Thus $f(r)$ is a smooth strictly positive radially symmetric function on $\mathbb{R}^3$. Finally (A3) implies $f(r) = C_1 r^{-3}$ for all $r \geqslant 1$, where C_1 is the strictly positive constant defined by

$$C_1^2 = -\int_0^1 t^4 (g^2)'(t)\,\mathrm{d}t.$$

Now define a smooth spinor field $\psi^{(1)}$ on $\mathbb{R}^3$ by

$$\psi^{(1)}(x) = f(r)\begin{pmatrix} x_3 \\ x_1 + \mathrm{i}x_2 \end{pmatrix} + \begin{pmatrix} \mathrm{i}g(r) \\ 0 \end{pmatrix}. \tag{3}$$

Thus

$$\langle \psi^{(1)}, \psi^{(1)} \rangle = r^2 f^2 + g^2. \tag{4}$$

Since $f(r) > 0$ for all $r \in \mathbb{R}^+$ and $g(0) = 2 > 0$, it follows that $\langle \psi^{(1)}, \psi^{(1)} \rangle$ is nowhere vanishing. On the other hand, using (1) we have

$$\langle \psi^{(1)}, \sigma\psi^{(1)} \rangle = \begin{pmatrix} 2x_1x_3 f^2 + 2x_2 fg \\ 2x_2x_3 f^2 - 2x_1 fg \\ (x_3^2 - x_1^2 - x_2^2) f^2 + g^2 \end{pmatrix}. \tag{5}$$

Therefore

$$\begin{aligned} \operatorname{div}\langle \psi^{(1)}, \sigma\psi^{(1)} \rangle &= 2x_3 f^2 + 2x_1^2 x_3 r^{-1}(f^2)' + 2x_1x_2 r^{-1}(fg)' \\ &\quad + 2x_3 f^2 + 2x_2^2 x_3 r^{-1}(f^2)' - 2x_1x_2 r^{-1}(fg)' \\ &\quad + 2x_3 f^2 + x_3 r^{-1}(x_3^2 - x_1^2 - x_2^2)(f^2)' + x_3 r^{-1}(g^2)' \\ &= x_3 r^{-5}(r^6 f^2)' + x_3 r^{-1}(g^2)' \\ &= 0 \end{aligned}$$

where the last line follows from (2).

Let $A^{(1)}$ be the vector potential given by proposition 1 with $\psi = \psi^{(1)}$. Now suppose $r \geqslant 1$. Thus $f(r) = C_1 r^{-3}$ and $g(r) = 0$ so (3) and (5) become

$$\psi^{(1)} = C_1 r^{-3} \begin{pmatrix} x_3 \\ x_1 + \mathrm{i}x_2 \end{pmatrix} \tag{6}$$

and

$$\langle \psi^{(1)}, \sigma\psi^{(1)} \rangle = C_1^2 r^{-6} \begin{pmatrix} 2x_1x_3 \\ 2x_2x_3 \\ x_3^2 - x_1^2 - x_2^2 \end{pmatrix}$$

respectively. Straightforward calculations then give

$$\operatorname{curl}\langle \psi^{(1)}, \sigma\psi^{(1)} \rangle = -2C_1^2 r^{-6} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}$$

and

$$\langle \psi^{(1)}, \nabla\psi^{(1)} \rangle = C_1^2 r^{-6} \begin{pmatrix} -2x_1 - \mathrm{i}x_2 \\ \mathrm{i}x_1 - 2x_2 \\ -2x_3 \end{pmatrix}.$$

Combining these expressions we thus have

$$A^{(1)} = \frac{1}{\langle \psi^{(1)}, \psi^{(1)} \rangle} \left(\frac{1}{2}\operatorname{curl}\langle \psi^{(1)}, \sigma\psi^{(1)} \rangle + \operatorname{Im}\langle \psi^{(1)}, \nabla\psi^{(1)} \rangle \right) = 0$$

on $\{r \geqslant 1\}$. Therefore $A^{(1)}$ is supported on the unit ball $\{r \leqslant 1\}$. Now $\psi^{(1)}$ is smooth whilst (6) implies $\langle \psi^{(1)}, \psi^{(1)} \rangle = C_1^2 r^{-4}$ when $r \geqslant 1$. Thus $\psi^{(1)} \in L^2$. Using similar reasoning it is in fact clear that $\psi^{(1)} \in H^{p,s}$ for any $p \in (3/2, +\infty]$ and $s \in \mathbb{R}$.

4. Example 2

Since $\operatorname{Ran}(g) = [0, 2]$ we have $4 - g^2 \geqslant 0$. Thus we can define a function h by

$$h(t) = \begin{cases} -(4 - g^2(t))^{1/2} & \text{for} \quad t \leqslant 0 \\ (4 - g^2(t))^{1/2} & \text{for} \quad t \geqslant 0. \end{cases}$$

Property (A2) of g gives $h(t) = t$ for $t \in [-1/2, 1/2]$. Since (A1), (A2) and (A4) imply $4 - g^2$ is strictly positive away from 0 it follows that h is smooth. Furthermore it is easy to check that

$$h^2 + g^2 = 4 \tag{7}$$

and

$$h(t) = \pm 2 \qquad \text{for} \quad \pm t \geqslant 1. \tag{8}$$

Using h, define a further smooth function k on $\mathbb{R}$ by $k = \int_0 h$. Now consider g, h and k as smooth functions on $\mathbb{R}^3$ depending on x_3 and constant in x_1 and x_2. Define a smooth function u and a smooth spinor field $\psi^{(2)}$ on $\mathbb{R}^3$ by

$$u = \mathrm{e}^{-\rho^2/2-k} \qquad \text{and} \qquad \psi^{(2)} = u \begin{pmatrix} (\rho^2-1)(-h+\mathrm{i}g) \\ 2(x_1+\mathrm{i}x_2) \end{pmatrix}.$$

By (8) we have $k = \pm 2x_3 + C_{2\pm}$ for $\pm x_3 \geqslant 1$, where $C_{2\pm}$ is some constant. It follows that u and hence $\psi^{(2)}$ are Schwartz class functions.

Now, using (7),

$$\langle \psi^{(2)}, \psi^{(2)} \rangle = ((\rho^2-1)^2(h^2+g^2) + 4\rho^2)u^2 = 4((\rho^2-1)^2 + \rho^2)u^2. \tag{9}$$

Since u is nowhere zero the same must be true for $\langle \psi^{(2)}, \psi^{(2)} \rangle$. Using (1) and (7) again we also get

$$\begin{aligned}\langle \psi^{(2)}, \sigma\psi^{(2)} \rangle &= 4u^2 \begin{pmatrix} (\rho^2-1)(-x_1h+x_2g) \\ (\rho^2-1)(-x_2h-x_1g) \\ (\rho^2-1)^2-\rho^2 \end{pmatrix} \\ &= -hv \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} + gv \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 4(\rho^4-3\rho^2+1)u^2 \end{pmatrix}\end{aligned}$$

where $v = 4(\rho^2-1)u^2$. A straightforward calculation now gives

$$\begin{aligned}\operatorname{div}\langle \psi^{(2)}, \sigma\psi^{(2)} \rangle &= -2hv - (x_1^2\rho^{-1} + x_2^2\rho^{-1})h\partial_\rho v - (x_1x_2\rho^{-1} - x_1x_2\rho^{-1})g\partial_\rho v \\ &\quad +4(\rho^4-3\rho^2+1)\partial_3(u^2) \\ &= -2hv - \rho h\partial_\rho v + 4(\rho^4-3\rho^2+1)\partial_3(u^2).\end{aligned}$$

However, $\partial_3(u^2) = -2hu^2$ whilst

$$\begin{aligned}-2hv - \rho h\partial_\rho v &= -8(\rho^2-1)hu^2 - 4\rho(2\rho + (\rho^2-1)(-2\rho))hu^2 \\ &= 8(\rho^4-3\rho^2+1)hu^2.\end{aligned}$$

The above expressions combine to give $\operatorname{div}\langle \psi^{(2)}, \sigma\psi^{(2)} \rangle = 0$. Let $A^{(2)}$ be the real vector potential given by proposition 1 with $\psi = \psi^{(2)}$; that is,

$$A^{(2)} = \frac{1}{\langle \psi^{(2)}, \psi^{(2)} \rangle}\left(\frac{1}{2}\operatorname{curl}\langle \psi^{(2)}, \sigma\psi^{(2)} \rangle + \operatorname{Im}\langle \psi^{(2)}, \nabla\psi^{(2)} \rangle\right).$$

A tedious, although straightforward, calculation allows us to compute $A^{(2)}$ explicitly from this formula. Firstly

$$\operatorname{curl} hv \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = 4(\rho^2-1)(h'-2h^2)u^2 \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix}$$

$$\operatorname{curl} gv \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix} = 4(\rho^2-1)(g'-2hg)u^2 \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 8(\rho^4-3\rho^2+1)gu^2 \end{pmatrix}$$

and

$$\operatorname{curl}\begin{pmatrix} 0 \\ 0 \\ 4(\rho^4 - 3\rho^2 + 1)u^2 \end{pmatrix} = -8(\rho^4 - 5\rho^2 + 4)u^2 \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix}.$$

On the other hand

$$\partial_1 \psi^{(2)} = -x_1 \psi^{(2)} + u \begin{pmatrix} 2x_1(-h + \mathrm{i}g) \\ 2 \end{pmatrix}$$

$$\partial_2 \psi^{(2)} = -x_2 \psi^{(2)} + u \begin{pmatrix} 2x_2(-h + \mathrm{i}g) \\ 2\mathrm{i} \end{pmatrix}$$

and

$$\partial_3 \psi^{(2)} = -h \psi^{(2)} + u \begin{pmatrix} (\rho^2 - 1)(-h' + \mathrm{i}g') \\ 0 \end{pmatrix}$$

so

$$\operatorname{Im} \langle \psi^{(2)}, \nabla \psi^{(2)} \rangle = u^2 \begin{pmatrix} -4x_2 \\ 4x_1 \\ (\rho^2 - 1)^2 (h'g - hg') \end{pmatrix}.$$

Combining the above calculations then gives

$$\begin{aligned} \langle \psi^{(2)}, \psi^{(2)} \rangle A^{(2)} = {} & 2(\rho^2 - 1)(g' - 2hg)u^2 \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \\ & + (2(\rho^2 - 1)(h' - 2h^2) - 4(\rho^4 - 5\rho^2 + 5))u^2 \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix} \\ & + u^2 \begin{pmatrix} 0 \\ 0 \\ 4(\rho^4 - 3\rho^2 + 1)g + (\rho^2 - 1)^2 (h'g - hg') \end{pmatrix}. \end{aligned}$$

Now, by (7),

$$2(\rho^2 - 1)(h' - 2h^2) - 4(\rho^4 - 5\rho^2 + 5) = 2(\rho^2 - 1)(h' + 2g^2) - 4((\rho^2 - 1)^2 + \rho^2).$$

From (9) we therefore get $A^{(2)} = (-x_2, x_1, 0)^{\mathrm{T}} + \tilde{A}^{(2)}$ where

$$\begin{aligned} 4((\rho^2 - 1)^2 + \rho^2)\tilde{A}^{(2)} = {} & 2(\rho^2 - 1)(g' - 2hg) \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} + 2(\rho^2 - 1)(h' + 2g^2) \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix} \\ & + \begin{pmatrix} 0 \\ 0 \\ 4(\rho^4 - 3\rho^2 + 1)g + (\rho^2 - 1)^2 (h'g - hg') \end{pmatrix}. \end{aligned}$$

Now, as functions on $\mathbb{R}^3$, $\operatorname{supp}(g)$, $\operatorname{supp}(h') \subseteq \{|x_3| \leqslant 1\}$ by (A3) and (8), respectively. It follows that $\tilde{A}^{(2)}$ is also supported on $\{|x_3| \leqslant 1\}$. Furthermore $\tilde{A}^{(2)}$ is bounded on this set. If we set

$$B^{(2)} = \operatorname{curl} A^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \tilde{B}^{(2)}$$

(so $\tilde{B}^{(2)} = \operatorname{curl} \tilde{A}^{(2)}$) then $\operatorname{supp}(\tilde{B}^{(2)}) \subseteq \{|x_3| \leqslant 1\}$ and

$$\tilde{B}_1^{(2)}, \tilde{B}_2^{(2)} = \mathrm{O}(\rho^{-1}) \qquad \tilde{B}_3^{(2)} = \mathrm{O}(\rho^{-2})$$

as $\rho \to +\infty$, uniformly for $|x_3| \leqslant 1$. In particular, $\tilde{B}^{(2)}$ decays at infinity.

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